4301. Use integration by parts twice. In the first, set $u = (\ln x)^2$. In the second, set $u = \ln x$.

4302. Set up the generic equation of the trajectory, by eliminating t from horizontal and vertical *suvat*s. Use an identity to write this as a quadratic in $\tan\theta$. Set $\Delta = 0$.

4303. Set up a polynomial f of odd degree. Show that the equation $\mathrm{f}''(x) = 0$ must have at least one root.

4304. The sum of a set of n independent normal variables is normal, with the variance equal to the sum of individual variances:

$$\sum_{i=1}^{n} X_i \sim N(0, n).$$

Consider the z value, as calculated by the formula

$$z = \frac{x - \mu}{\sigma}.$$

4305. Sum the integers from 1 to ab. Then subtract the total of the integers which are divisible by a. Each of these totals is the sum of an AP.

4306. As is often the case with sigma notation, the result is easier to understand written longhand. Because you don't know that the sum to infinity converges, you need to work with a limit: the definition of an infinite sum is

$$S_\infty = \lim_{n\to\infty} S_n.$$

4307. (a) Make $\cos s$ and $\sin s$ the subjects. Square your equations and add them, simplifying with the first Pythagorean trig identity.
(b) Consider $\mathrm{f}(t)$ and $\mathrm{g}(t)$ as (length) scale factors in the x and y directions.

4308. (a) A geometric series only converges for $|r| < 1$.
(b) Work out the infinite sum using

$$S_\infty = \frac{a}{1 - r}.$$

Then consider graphically/sketch the function over the domain from (a).

4309. Since $y = x^{2p}$ has $x = 0$ as a line of symmetry and $y = x^{2q+1}$ has the origin as a centre of rotational symmetry, any reflective symmetry can only be in $x = 0$, and any rotational symmetry can only be around O. Algebraically, these symmetries are

$$\begin{aligned} \text{Even} &: \mathrm{f}(-x) = \mathrm{f}(x), \\ \text{Odd} &: \mathrm{f}(-x) = -\,\mathrm{f}(x). \end{aligned}$$

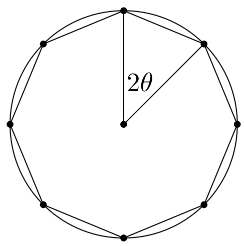
Test for each.

4310. The centres of the cannonballs are equidistant, so they form a regular tetrahedron. Let this have side length 1. Find the angle of inclination of each of the sloped edges. Resolve vertically, calculating the vertical component of each of each of the three (symmetrical) reaction forces.

4311. Use a combinatorial approach, counting outcomes. You can count successful outcomes on the fingers of one hand.

4312. Firstly, combine the integrals and factorise. Then use the second Pythagorean trig identity. You can then integrate by inspection (the reverse chain rule). The substitution $u = \tan x$ would also work, although it's a long way round.

4313. (a) Without loss of generality, take $(1, 0)$ as the initial position. The final position is then $(\cos 2\theta, \sin 2\theta)$.
(b) Use a double-angle formula.
(c) One full circle is 2π radians.
(d) The scenario is as follows, with 2θ subtended by each sector.



As $\theta \to 0$, the ratio of the perimeter of the polygon to the circumference of the circle tends to 1.

4314. Let $u = (\ln x)^2$ and $\frac{dv}{dx} = 1$.

4315. Let $z = \sec x$ and $y = \tan x$.

4316. For f to be well defined, the radicand $x^4 + x^2 - 6$ must be positive.

4317. No calculation is needed. You can write the answer down.

4318. Solve for intersections, substituting for x^2. You should find that there are up to four. Proceed case by case. You need the first quadratic in y to have at least one root. Consider the cases in which it has exactly one root, or two roots. Then look at the sign of these roots to see whether they produce x values.

4319. Since a is close to zero, you can use small-angle approximations. You'll need the formulae for both sine and cosine.

4320. (a) Draw a force diagram, making sure you define x, T and a in consistent directions.
(b) Find $\ddot{x}$ for the proposed solution, and sub into the DE.
(c) Consider the combination of the two terms in harmonic form.

4321. Find the equation of a generic tangent line. Set up an equation solving for intersections of this with $y = x^3$. You know that this equation has a root at $x = p$. Take the associated factor out and consider the remaining quadratic.

4322. Use the formula $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$ to test the independence of A and B.

4323. (a) Use a small-angle approximation.
(b) The domain is too large to use a small-angle approximation. So, find the range of f over the given domain. If the range, which has the form $[-k, k]$, is sufficiently narrow, then the function could be approximated with zero.

4324. Look for intersections, reaching the equation

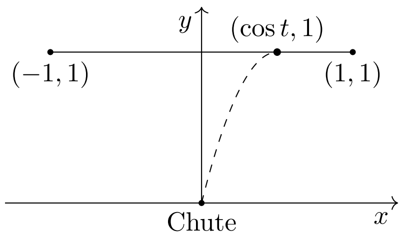
$$y^4 - y^2 - 2y + k = 0.$$

You need this to have a repeated root, so you need the function $\text{f}(x) = y^4 - y^2 - 2y + k$ to have a stationary point (which is also a root). Set the derivative to zero. This allows you to get rid of k.

4325. Separate the variables by first adding $\frac{t}{P}$ to both sides of the equation. Then integrate, finding the $+c$ using the initial conditions. Then sub $t = 10$.

4326. Use the factor theorem on $\text{f}(x) - \text{g}(x) = 0$.

4327. The scenario, with a trajectory after release shown as a dashed line, is



Find the position and velocity at time t_0. Then use these as initial conditions of projectile motion, which is required to pass through the origin.

4328. Find the area of an intersection, by considering it as two segments subtending a right angle at the centre. Then add up the areas of the circles and subtract one copy of the intersections (they have been counted twice).

4329. (a) Differentiate by the chain rule.
(b) Set the derivatives to zero, and consider a quadratic in e^x.
(c) The curves have odd and even symmetry. Also, both tend to $y = \frac{1}{2}e^x$ as $x \to \infty$. Their behaviours differ as $x \to -\infty$.

4330. (a) Solve $k \displaystyle\int_0^1 x - 2x^2 + x^3 \, dx = 1$.
(b) Set up a generalised version of the equation in part (a). This time, integrate by parts.

4331. The probability distribution of $B(5, 1/3)$ is

x	0	1	2	3	4	5
$\mathbb{P}(X = x)$	$\frac{32}{243}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{40}{243}$	$\frac{10}{243}$	$\frac{1}{243}$

Use the formula $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

4332. Find the domain and codomain of arcsec, which should be as broad as possible while keeping the function one-to-one. A sketch of the relevant graph will help, as ever. Then consider the effect of the input transformation $x \mapsto 2x + 1$ on this domain and codomain/range.

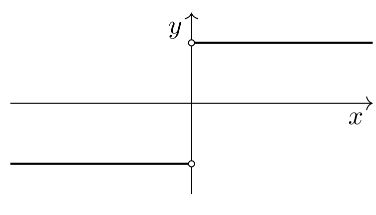
4333. (a) Set up horizontal and vertical *suvat*s. Sub for t and use the second Pythagorean trig identity.
(b) Solve the quadratic.
(c) Calculate the gradient of the trajectory at the hoop for both angles in (b). You need a (large) negative gradient.

4334. You don't need to use any algebra or coordinate geometry here. Consider rotational symmetry.

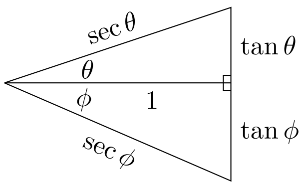
4335. Integrate by parts twice. If you know it, use the tabular integration method.

4336. Differentiate implicitly, and set $\frac{dy}{dx} = 0$ without making it the subject. Substitute this back into the curve, and use a polynomial solver to find a root. Take out the relevant factor and show that what remains has no real roots. Then substitute the x value back in and find y.

4337. The graph of the function $\text{f}(x) = \frac{|x|}{x}$ is



4338. Two of the side lengths are $\sec\theta$ and $\sec\phi$, and the third is $(\tan\theta + \tan\phi)$.



4339. You can count the list the successful outcomes. Consider only runs of heads, classifying by length of run $4, 5, 6$.

4340. For the particle to come to rest, both components of velocity must be zero. Show, by solving each individually, that no value of t satisfies both.

4341. This just requires a bit of algebraic slog. Multiply up by the denominators. Take the LHS and RHS separately, simplifying the sextics. Subtract these, take out a factor of x and use a polynomial solver to deal with the remaining quartic.

You might cancel $(x-1)$ in the RH fraction before you start, although it doesn't simplify things all that much. If you don't, keep watch for how that non-cancellation crops up later.

4342. Both techniques work.

4343. The curve is symmetrical, so the point of tangency in the positive quadrant lies on the line $y = x$. Solve this simultaneously with the curve.

4344. (a) The sum can be represented as a rectangular approximation to the integral given, with the upper-left corners of the rectangles placed on the curve $y = \frac{1}{x}$.

(b) Calculate the integral, in the limit as $n \to \infty$. Note that the range of ln is $\mathbb{R}$.

4345. Multiply out, and use $|x|^2 \equiv x^2$.

4346. Neither inequality holds in general, although each often does. Find counterexamples with equality.

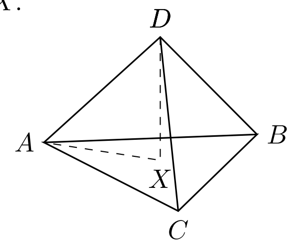
4347. (a) Take natural logs of both sides of the proposed relationship, and rearrange to a linear form.

(b) The points should be very close to collinear.

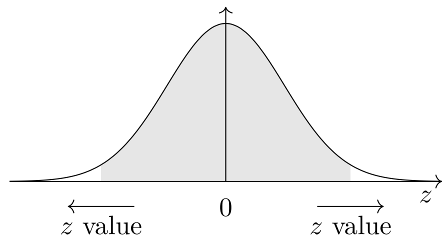
(c) Compare the equation of your line of best fit to the linear relationship in part (a), calculating the value of k.

4348. Lay the division out as a multiplicative equation, in the manner that "$13 \div 5 = 2$ remainder 3" can be expressed as $13 = 2 \times 5 + 3$. Then substitute the value $x = \alpha$.

4349. Let the side length of the tetrahedron be 1. Drop a perpendicular to the base ABC at point X, and solve $\triangle DAX$.



4350. Let the population have distribution $N(\mu, \sigma^2)$. Show algebraically that the z values associated with the probability in question tend as follows:



4351. (a) Factorise the RHS of the graph, so as to find its roots and the multiplicity of its roots. This will allow you to put axes onto the profile. Set up a definite integral to calculate the cross-sectional area on the profile, and multiply by the width of the stream.

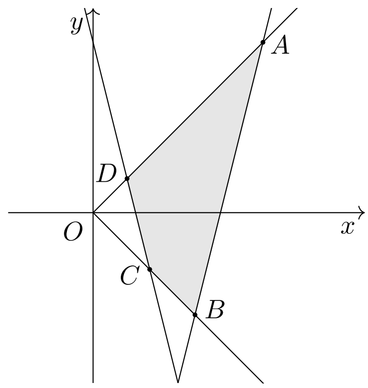
(b) Locate the relevant SP and find the equation of the surface of the upper pool in the form $z = k$. Set up an equation to find the LH boundary of the upper pool. This is a quintic, so use N-R. Set up an integral as before.

4352. Calculate the angles of inclination of the lines.

4353. Three of the curves have intersections, two don't. You can answer the question by subtracting e.g. 96 from all of the indices.

4354. Let $x = \arcsin t$, so $t = \sin x$. Write $\tan x$ in terms of t. Then integrate by inspection.

4355. Graphs:

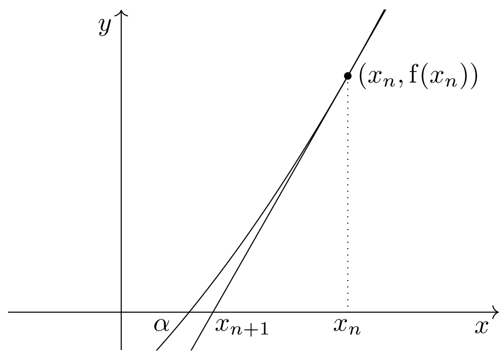


Find the area of $\triangle OAB$, and subtract the area of triangle $\triangle OCD$.

4356. (a) Just plug the numbers in.
    (b) And again.
    (c) The number of new infections is the definite integral of the rate at which infections occur.

4357. (a) Show that the numerator has no roots.
    (b) Differentiate by the quotient rule.
    (c) Consider the quadruple asymptote at $x = 0$, and the even symmetry of the graph.

4358. The N-R iteration generates a new approximation x_{n+1} to a root α at the x intercept of the tangent line at x_n. Use the following diagram:



4359. (a) Draw a force diagram for the bar, without the moment applied by the hinge. It isn't possible to show this moment as a regular force arrow, since it is modelled as acting at (a negligible distance from) the hinge, which is the pivot point.
    (b) Draw a force diagram for the bob, splitting the contact force into R_x, R_y. Resolve horizontally and vertically. Note that the acceleration of the bob must be perpendicular to the bar.

4360. A conditioning approach is slightly easier than a combinatorics approach here. Place A_1 without loss of generality. With probability $\frac{2}{5}$, A_2 leaves a gap of one; with probability $\frac{1}{5}$, A_2 sits opposite. Address these case by case:

    ① $A_1{*}A_2{*}{*}{*}$
    ② $A_1{*}{*}A_2{*}{*}$

4361. Use the compound-angle formula

$$\tan(x+h) \equiv \frac{\tan x + \tan h}{1 - \tan x \tan h}.$$

Use the small-angle approximation $\tan h \approx h$ and the second Pythagorean trig identity.

4362. In plain English, the question is asking: "Find the point inside the triangle which is furthest away from the vertices."

4363. Firstly, choose the $r_1 + r_2$ squares on which the counters are to go. Then, choose from them the r_1 squares on which the black counters are to go.

4364. The first equation is a quadratic in xy.

4365. Compare 2D and 3D.

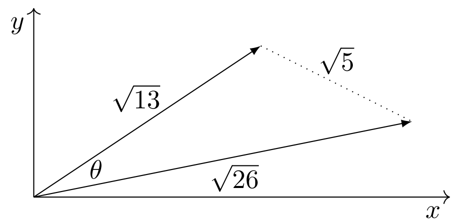
4366. Use $\log_y x \equiv \frac{1}{\log_x y}$ to set up a quadratic in $\log_x y$. Solve, then consider the two graphs that emerge. You only need look at the +ve quadrant.

4367. The three probabilities, in ascending rather than question order, are $p = 0, \frac{1}{2}, 1$.

4368. Sketch the boundary equations first, then set up a definite integral for the area.

4369. Start with $\mathrm{f}''(x) = \mathrm{g}''(x)$ and integrate twice.

4370. Use the cosine rule in the following diagram:



4371. Firstly, sketch $y = x^2 - 2|x|$. Then reciprocate.

4372. The graphs are symmetrical in $y = x$: the centre must lie on $y = x$. Use the fact that tangent and radius are perpendicular (and calculus) to find the point of tangency with $4y = x^2$.

4373. Set up the equation $nx^2 + (n+1)x + (n+2) = 0$. Then consider the inequality $\Delta \geq 0$.

4374. (a) Set up a vertical *suvat* with zero final velocity.
    (b) Find the time of flight and then the range, in terms of u, θ and g. Then use the result of part (a) to get rid of θ.

4375. The number of possible hands is ${}^{52}\mathrm{C}_5$. For a full house, there are 13 choices of number for three of a kind and then 12 choices of number for the pair. Once the numbers have been chosen, e.g. three sixes and two queens, work out the number of way in which the individual cards can be chosen.

— Alternative Method —

For a conditioning approach, the cards must be AAABB in some order. Work out the probability of getting AAABB in that order. Then multiply by the number of orders of AAABB.

4376. Sketch the graph $y = \operatorname{arccot} x$ first, then consider the symmetry of $y = \operatorname{arccot}\left(x^2\right)$.

4377. (a) Use the cosine rule, and then use a small-angle approximation.

(b) In the limit, A, B, C become collinear.

4378. Start as follows:

The expansions in the numerator are

$$(ax \pm h)^n \equiv a^n x^n \pm n a^{n-1} x^{n-1} h + \ldots.$$

The expansions in the denominator are

$$(bx \pm h)^n \equiv b^n x^n \pm n b^{n-1} x^{n-1} h + \ldots.$$

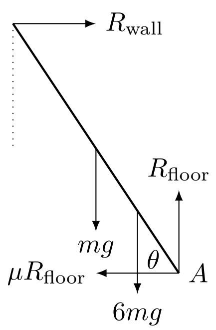
So, the numerator is...

4379. You can reverse-engineer (and then *show* forwards) the solution by factorising the denominator.

4380. Sketch the three boundary equations. The first two are easy. The third, $x^3 + y^3 = 1$, is more challenging, but can be seen as follows:

- In the positive quadrant, the behaviour is akin to $x^2 + y^2 = 1$.
- There are no points in the negative quadrant.
- In the other two quadrants, the curve is asymptotic to $x + y = 0$.
- The tangent is parallel to the x axis at the y intercept and parallel to the y axis at the x intercept

4381. The force diagram, assuming limiting friction, is



4382. Show that the equation of the normal is

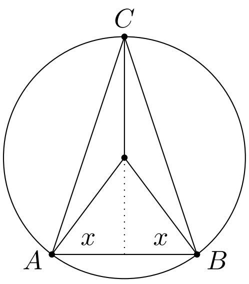
$$y = -\tfrac{1}{2p}x + p^2 + \tfrac{1}{2}.$$

Solve simultaneously with $y = x^2$. The resulting quadratic in x should give two roots $x = p$ and another, which is then $x = q$. Set up and simplify $q^2 - p^2$.

4383. This is a quadratic in $a^{\frac{1}{3}}$.

4384. (a) Consider, using $A_\triangle = \frac{1}{2}bh$, the perpendicular bisector of AB as a line of symmetry.

(b) The scenario is now



Find the length of the dotted perpendicular and use $A_\triangle = \frac{1}{2}bh$.

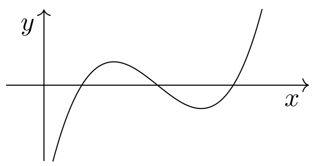
(c) For optimisation, set

$$\frac{d}{dx}\big(A_\triangle\big) = 0.$$

Show that $A_\triangle$ is stationary when $x = \frac{\sqrt{3}}{2}r$.

4385. For f to be invertible over $\mathbb{R}$, it must be be one-to-one. So, it can have no turning points. Find all values of a and b such that there is no sign change in the gradient.

4386. Since the roots are in AP, the central root must be the point of inflection of $y = 8x^3 - 36x^2 + 46x - k$, which must be on the x axis.



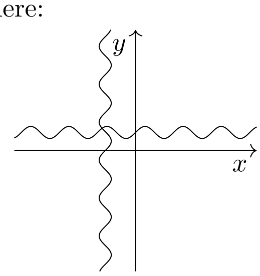
4387. (a) The derivative of $\sec x$ is $\sec x \tan x$. This is a (relatively) standard result you'll need in (b).

(b) The LHS should take the form $\big(y\,\mathrm{f}(x)\big)'$.

(c) Integrate the result of (b).

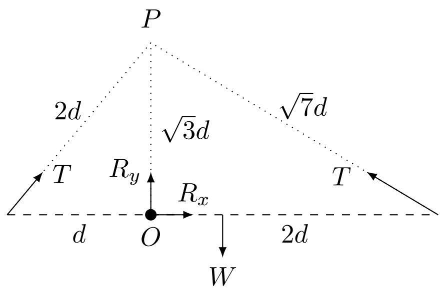
4388. Use a combinatorial (or a conditioning) approach. For each, consider the two cases AAAB and AABB separately. The combinatorial approach sidesteps a tricky point in the case AABB.

4389. Consider the curves on a large scale: $y = \sin x + p$ is on average parallel to the x axis, and $x = \sin y + q$ is on average parallel to the y axis. So, they must cross somewhere:



Consider the range of the gradients to show that they cannot intersect more than once.

4390. Using Pythagoras to calculate various lengths, the force diagram for the bar is

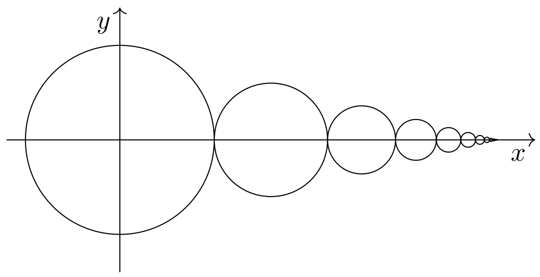


Take moments around O.

4391. Begin by considering the inequality $x^2 < x^3$. Solve this, then you can restrict things to a manageable domain. Show that, over such a domain, x^2 can not be less than $\sin x$. Sketching graphs will (of course!) help.

4392. By considering the sign change of $x + 1$, you can ignore the mod sign. Integrate by parts. When substituting in the infinite limit, consider the fact that exponential decay is more powerful than any polynomial growth.

4393. Since the total length is finite, the GP must be decreasing. No circle lies inside another, so the circles are tangent to one another in an increasing sense, i.e. one after another along the x axis. The scenario, not to scale, is as follows:



4394. (a) To remain on a single parabola, the projectile must meet the surfaces at right angles.

(b) Use the initial velocity (you know its direction) to set up horizontal and vertical *suvat*s. Then eliminate t to get the equation of trajectory, using h as the y intercept. Differentiate this and use the fact in part (a).

4395. (a) Solve!

(b) For infinitely many (x, y) solution points, the equations must be the same.

4396. You don't need to use calculus here, although you could. For a geometric approach, the quantity $2x + y$ is maximised in the direction of the line $y = \frac{1}{2}x$, which is normal to $2x + y = k$.

4397. Start with the LHS. Write it as a fraction involving $\cos\frac{1}{2}$ and $\sin\frac{1}{2}$ using compound-angle formulae. Then multiply top and bottom by the conjugate of the bottom. You can then use a Pythagorean and two double-angle formula to simplify.

4398. In this case (but not generally with parametrics) it is easier to find the Cartesian equation.

4399. Show that the (restricted) possibility space has 20 equally likely outcomes in it.

4400. (a) Integrate by parts twice. For this, the tabular integration method is easiest. Then substitute in the initial conditions to find the constant of integration.

(b) Factorise and use the discriminant.

—— End of 44th Hundred ——